MATH 303 – MEASURE THEORY HISTORY AND MOTIVATION

8 September 2025

THE PROBLEM OF MEASUREMENT

PROBLEM: COMPUTE THE SIZE (LENGTH, AREA, VOLUME) OF GEOMETRIC OBJECTS

- What are the "geometric objects" to which we want to (and are able to) assign a notion of size?
- What properties should size (length, area, volume) satisfy?
- How do we compute sizes of geometric objects?

WHAT ARE SOME "GEOMETRIC OBJECTS" WE MAY WANT TO ASSIGN A NOTION OF SIZE IN EUCLIDEAN SPACE?

CIRCLE

HOURGLASS

POLYGONS

FRACTALS

SPHERES

CIRCLE

SET

KLEIN BOTTLE

COMPACT SPACES

CURCLE, IRREGULAR SHAPE

WHAT ARE SOME PROPERTIES THAT "SIZE" SHOULD SATISFY?

BIG OBJECT -> BIG NUMBER

ADDITIVITY EMPTY SET SCALAR

POSITIVITY

TRANSLATION INVARIANCE

NON-NEGATIVITY

COMMUTATIVITY

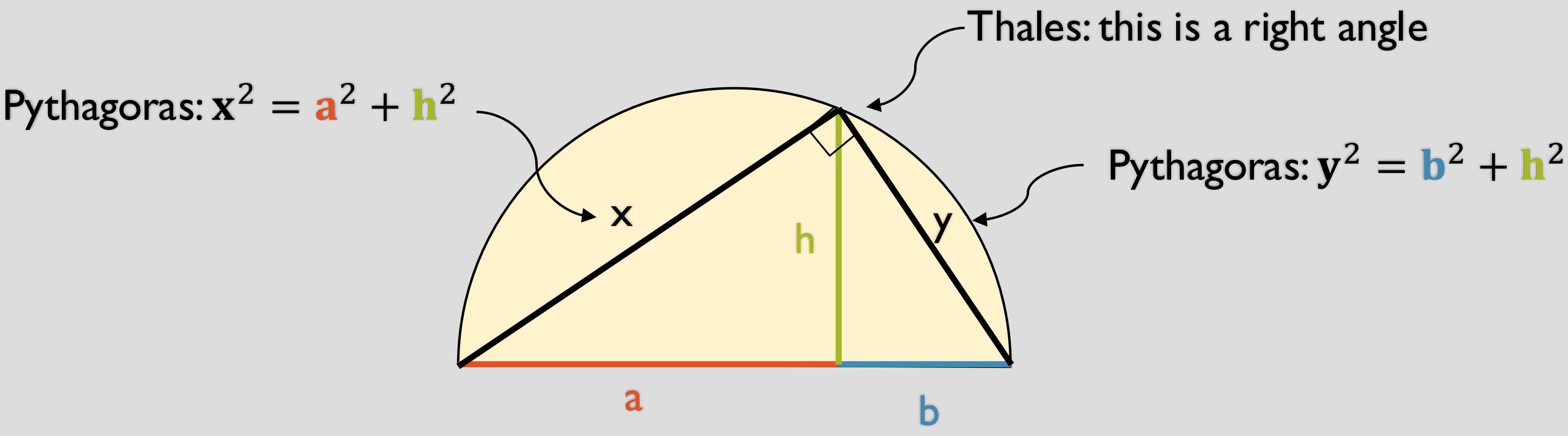
METHODS OF ANTIQUITY

Quadrature (squaring) and the method of exhaustion

SQUARING

- Computation of area by constructing a square (with straightedge and compass) of equal area to a given object
- Works well for polygons
- Does not work well for curved regions (Lindemann, 1882: “squaring the circle” is impossible)

EXAMPLE: SQUARING A RECTANGLE

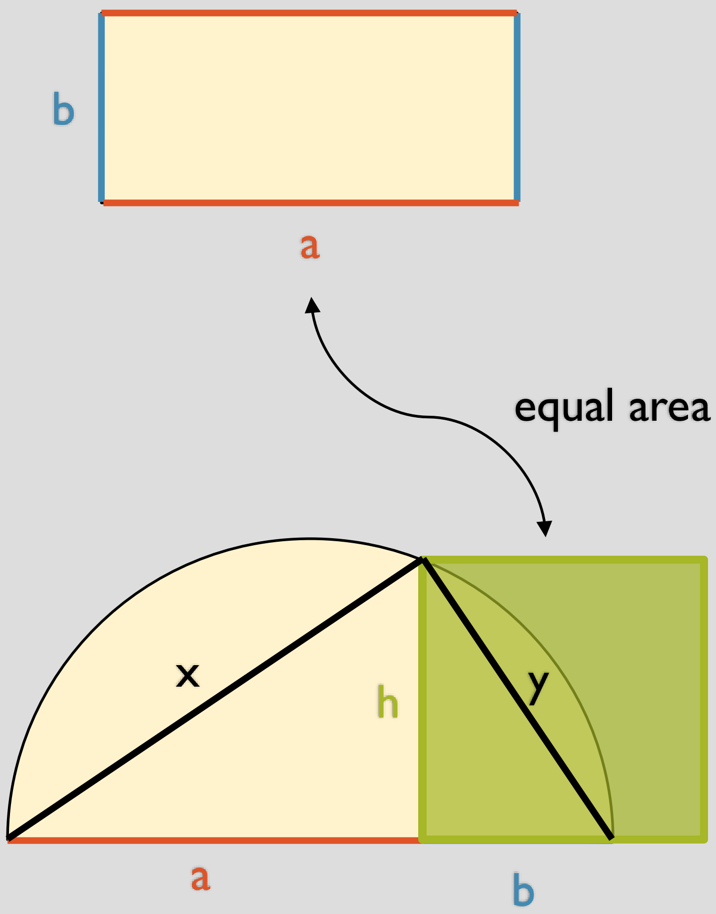


Pythagoras: $(a + b)^2 = x^2 + y^2$

$$x^2 = a^2 + h^2$$
$$y^2 = b^2 + h^2$$
$$(a + b)^2 = x^2 + y^2$$

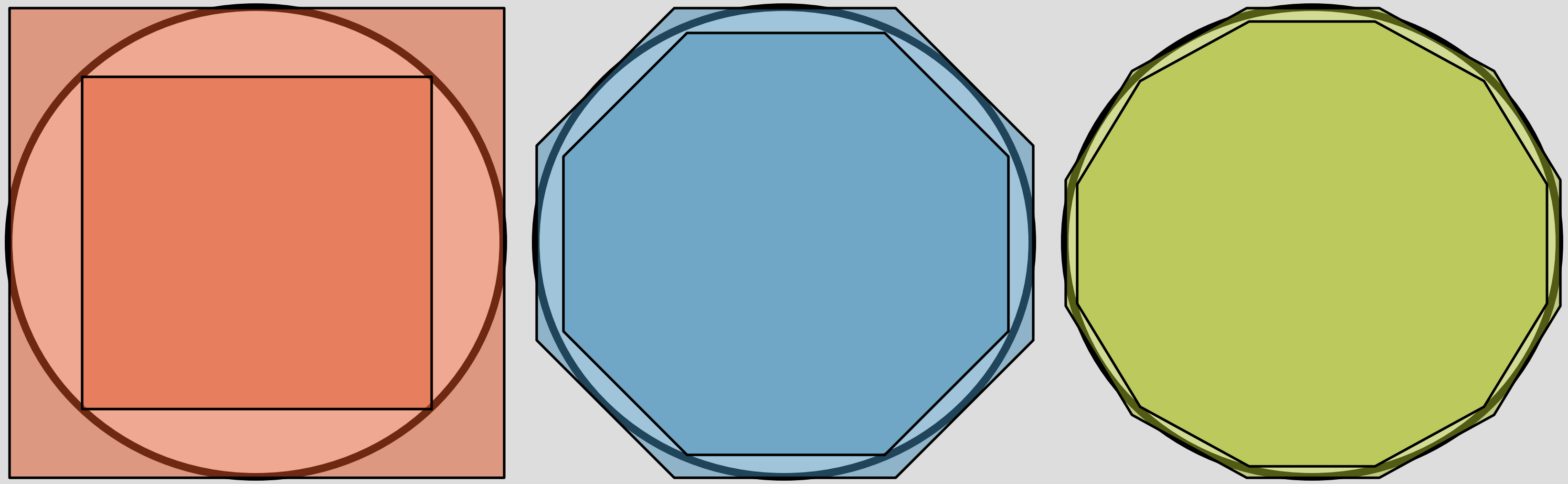
$$a^2 + b^2 + 2ab = a^2 + h^2 + b^2 + h^2$$

$$ab = h^2$$



METHOD OF EXHAUSTION

- Compute area as "limit" of inscribed/circumscribed polygons
- Formalized by Eudoxus
- Best-known method for computation of π until end of 17th century



INDIVISIBLES AND INFINITESIMALS

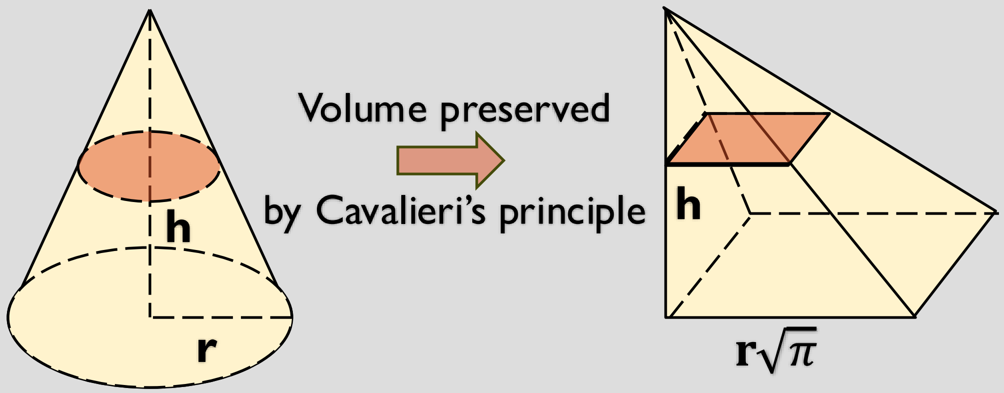
CAVALIERI'S PRINCIPLE

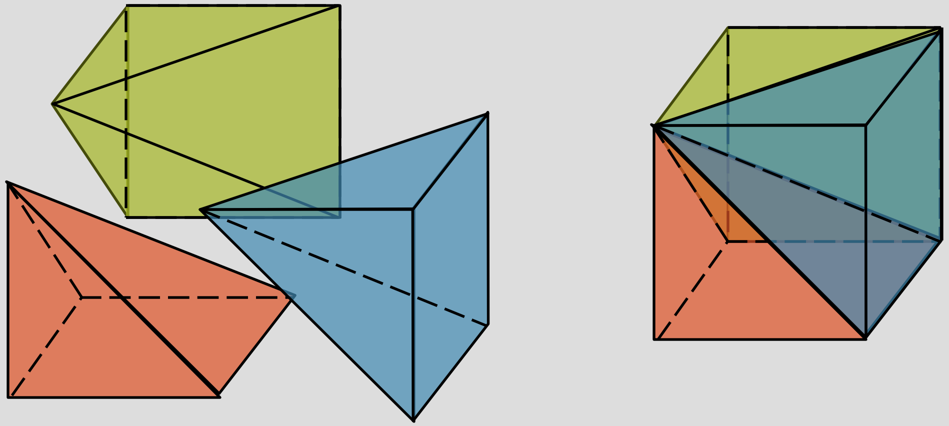
- Suppose two regions in the plane are bounded between two parallel lines.
- If the two regions have cross-sections of equal length, then they have equal area.
- Higher dimensional version holds for comparing volumes by means of cross-sectional areas

EXAMPLE: VOLUME OF A CONE

Cavalieri's Principle

Suppose two regions in space are bounded between two parallel planes. If the two regions have planar cross-sections of equal area, then they have equal volume.

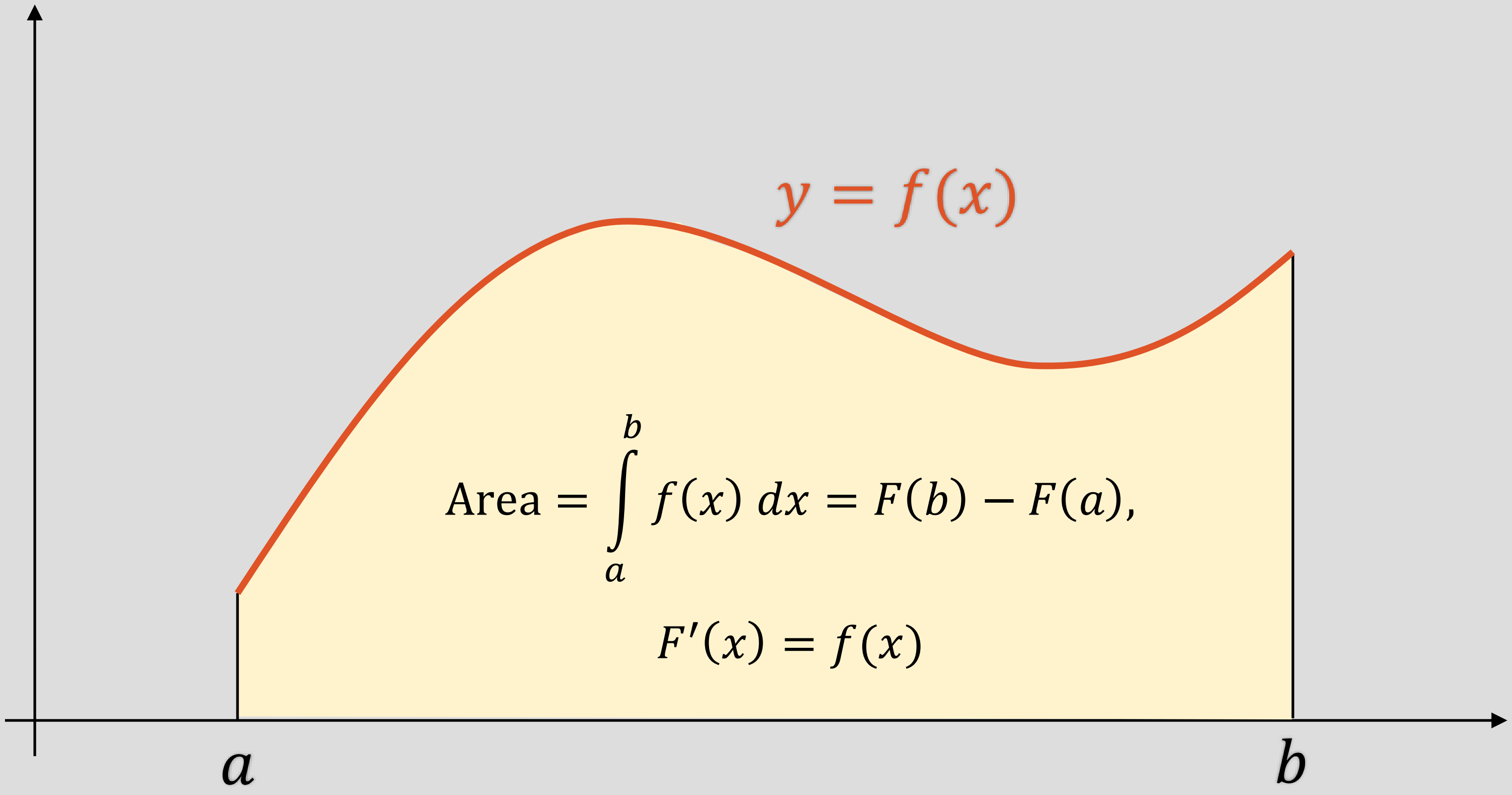




Square pyramid is 1/3 of a box → volume $= \frac{1}{3}\pi \mathbf{r}^2\mathbf{h}$

INTEGRAL CALCULUS

FUNDAMENAL THEOREM OF CALCULUS



ε AND δ

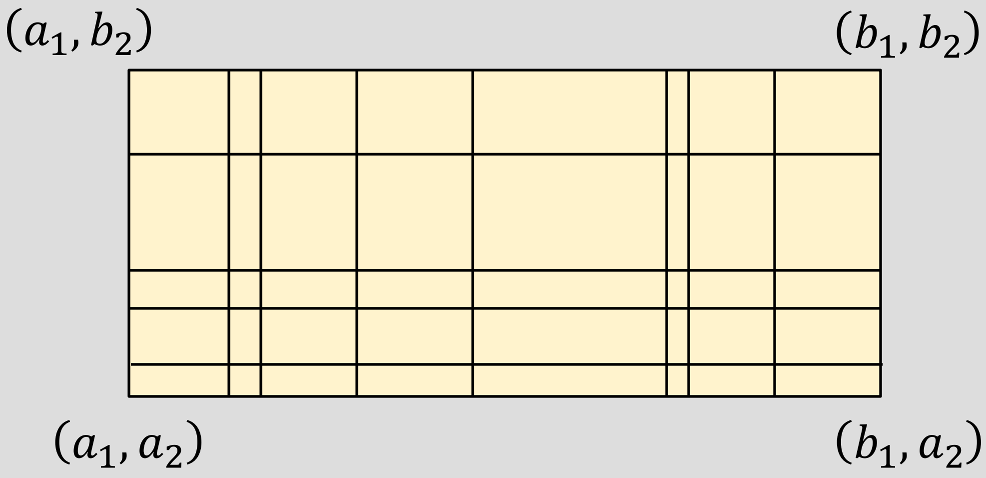
Rigorous Foundations for Calculus

RIEMANN—DARBOUX INTEGRATION

- ***Darboux partition*** of $B = \prod_{i=1}^{d}[a_i, b_i]$:

$$P = \left(x_{i,j}\right)_{1 \le i \le d, 0 \le j \le n_i}$$

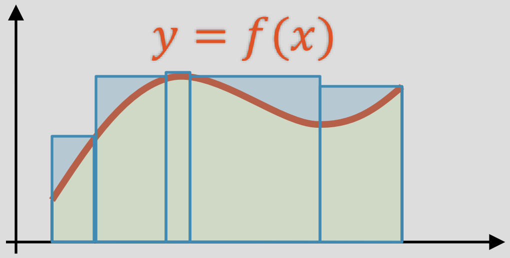
with $a_i = x_{i,0} < x_{i,1} < \cdots < x_{i,n_i} = b_i$

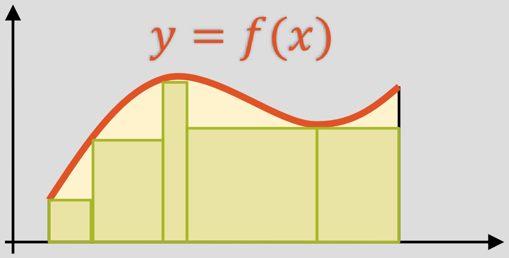


- ***Upper*** and ***lower Darboux sums*** of f:

$$U_B(f, P) = \sum_{j \in \prod_i \{1,\dots n_i\}} \sup_{x \in B_j} f(x) \cdot \mathrm{Vol}(B_j)$$

$$L_B(f, P) = \sum_{j \in \prod_i \{1,\dots n_i\}} \inf_{x \in B_j} f(x) \cdot \mathrm{Vol}(B_j)$$





RIEMANN—DARBOUX INTEGRATION

- ***Upper*** and ***lower Darboux integral*** of f:

$$U_B(f) = \inf\{U_B(f, P) : P \text{ is a Darboux partition of } B\}$$

$$L_B(f) = \sup\{L_B(f, P) : P \text{ is a Darboux partition of } B\}$$

- f is ***(Riemann—)Darboux integrable*** over B if $U_B(f) = L_B(f)$
- Denote integral by $\int_B f(x)\, dx$

JORDAN CONTENT

- A bounded set $E \subseteq \mathbb{R}^d$ is ***Jordan-measurable*** if 1_E is Riemann—Darboux integrable over a box containing E
- The ***Jordan content*** of (a Jordan-measurable set) E is $J(E) = \int_B 1_E(x)\, dx$

- ***Simple set***: finite union of boxes

$$S = \bigcup_{n=1}^{N} B_n$$

- ***Outer Jordan content***:
  $J^*(E) = \inf\{\mathrm{Vol}(S) : S \supseteq E \text{ simple}\}$
- ***Inner Jordan content***:
  $\mathrm{J}_*(E) = \sup\{\mathrm{Vol}(S) : S \subseteq E \text{ simple}\}$

JORDAN-MEASURABILITY

Theorem

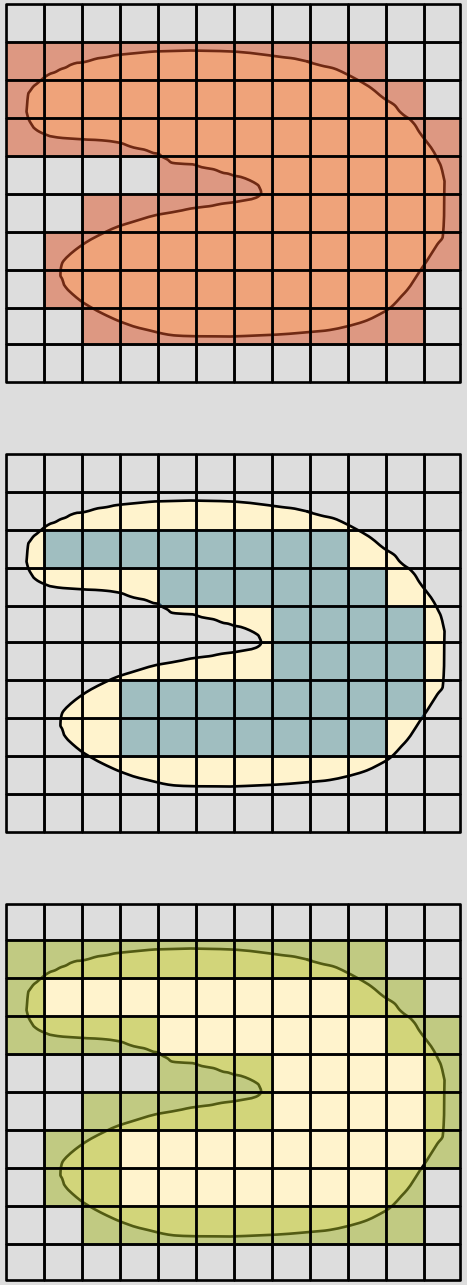
The following are equivalent for a bounded set $E \subseteq \mathbb{R}^d$:

i. E is Jordan-measurable

ii. $J^*(E) = J_*(E)$

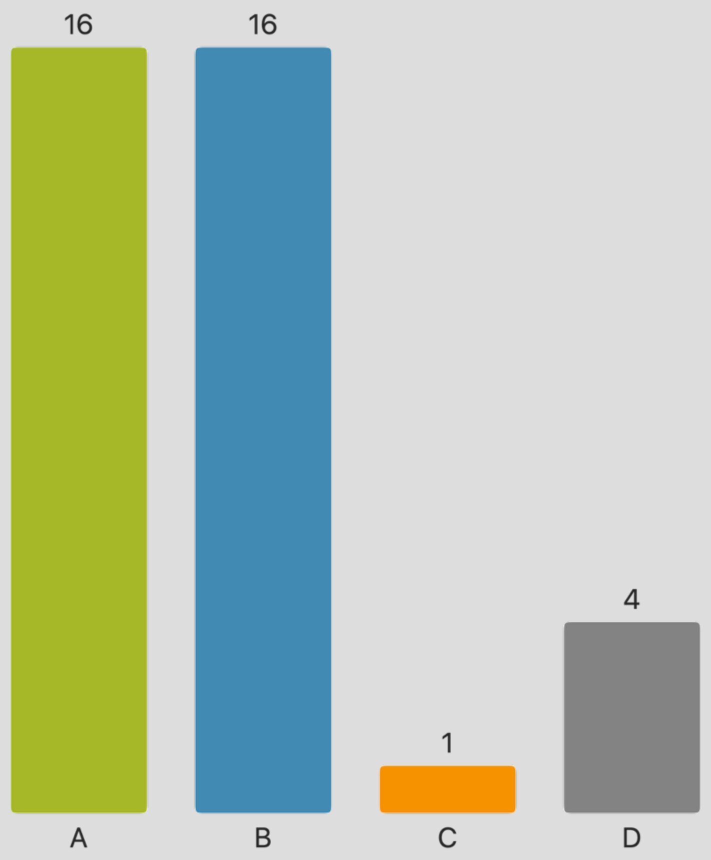
iii. $J^*(\partial E) = 0.$

Jordan-measurable ⇔ method of exhaustion converges



WHICH OF THE FOLLOWING SETS ARE JORDAN-MEASURABLE? (SELECT ALL THAT APPLY)

- ✓ A. Polygons
- ✓ B. Circles
- C. The set of rational numbers in [0,1]
- ✓ D. The middle thirds Cantor set



SET THEORY, CHOICE, AND THE IMPOSSIBILITY OF MEASURING EVERYTHING

WHAT ARE SOME PROPERTIES THAT "SIZE" SHOULD SATISFY?

- See responses from earlier

PROBLEM OF MEASUREMENT – SET THEORETIC FORMULATION

- Does there exist a function $\mathrm{Vol} : \mathcal{P}(\mathbb{R}^d) \to [0, \infty]$ such that
  - **Normalized**: $\mathrm{Vol}\left([0,1)^d\right) = 1$
  - **Isometry-invariant**: if A and B are isometric, then $\mathrm{Vol}(A) = \mathrm{Vol}(B)$
  - **Countably additive**: if $E_1, E_2, \ldots$ are pairwise disjoint, then

$$\mathrm{Vol}\left(\coprod_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mathrm{Vol}(E_n)$$

VITALI SETS

Vitali's Theorem

There is no normalized, translation-invariant, countably additive function defined on all subsets of the real line.

Normalized: $\mathrm{Vol}\left([0,1)^d\right) = 1$

Isometry-invariant: if A and B are isometric, then $\mathrm{Vol}(A) = \mathrm{Vol}(B)$

Countably additive: if $E_1, E_2, \ldots$ are pairwise disjoint, then

$$\mathrm{Vol}\left(\coprod_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mathrm{Vol}(E_n)$$

***Proof*.**

- Define equivalence relation on $[0,1)$:
$$x \sim y \Leftrightarrow x - y \in \mathbb{Q}$$
- By ***axiom of choice***, let V contain exactly one element from each equivalence class
- For $t \in [0,1)$, let
$$E_t = \{x + t \bmod 1 : x \in E\}$$
- The sets $(E_t)_{t \in \mathbb{Q} \cap [0,1)}$ are pairwise disjoint
- $\coprod_{t \in \mathbb{Q} \cap [0,1)} E_t = [0,1)$
- Suppose for contradiction that a function L exists
- By ***isometry-invariance***, $L(E_t) = L(E)$
- By ***countable additivity***,
$$L\big([0,1)\big) = \sum_{t \in \mathbb{Q} \cap [0,1)} L(E_t) = \infty \cdot L(E)$$
- By ***normalization***, $L\big([0,1)\big) = 1$
- ***Contradiction!***

WEAK PROBLEM OF MEASUREMENT

- Does there exist a function $\mathrm{Vol} : \mathcal{P}(\mathbb{R}^d) \to [0,\infty]$ such that
  - **Normalized**: $\mathrm{Vol}([0,1)^d) = 1$
  - **Isometry-invariant**: if A and B are isometric, then $\mathrm{Vol}(A) = \mathrm{Vol}(B)$
  - ***Finitely additive***: if A, B are disjoint, then
  $$\mathrm{Vol}(A \sqcup B) = \mathrm{Vol}(A) + \mathrm{Vol}(B)$$

IS THE WEAK PROBLEM OF MEASUREMENT SOLVABLE?

Weak Problem of Measurement

Does there exist a function $\mathrm{Vol} : \mathcal{P}(\mathbb{R}^d) \to [0, \infty]$ such that

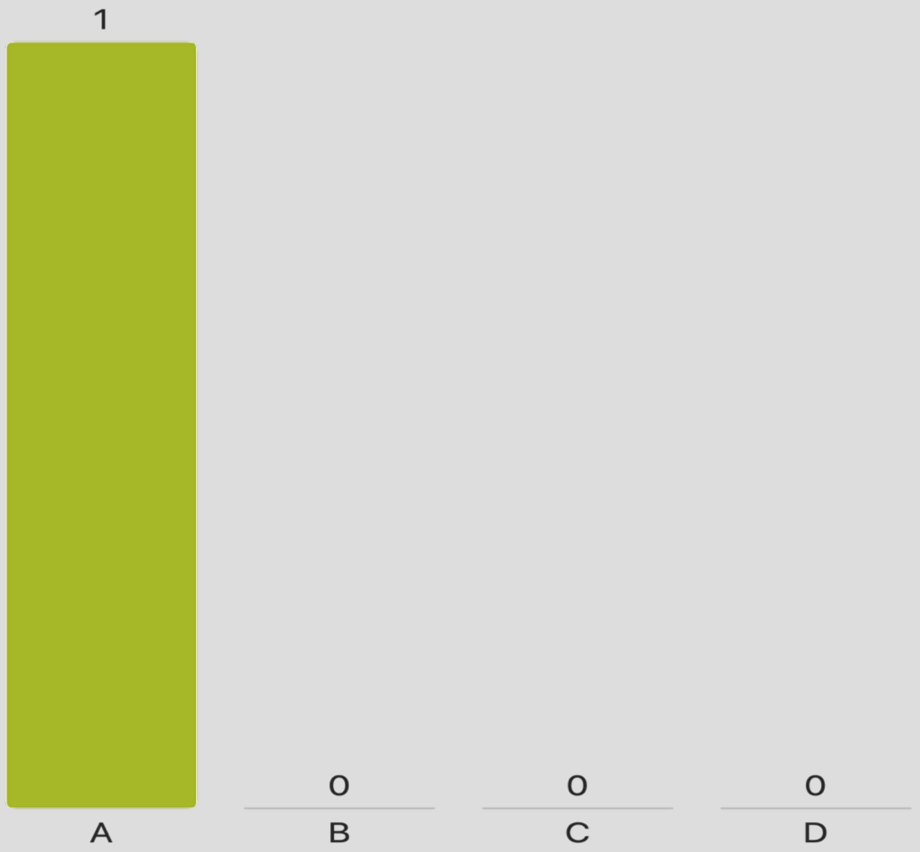
- **Normalized**: $\mathrm{Vol}([0,1)^d) = 1$
- **Isometry-invariant**: if A and B are isometric, then $\mathrm{Vol}(A) = \mathrm{Vol}(B)$
- **Finitely additive**: if A, B are disjoint, then $$\mathrm{Vol}(A \sqcup B) = \mathrm{Vol}(A) + \mathrm{Vol}(B)$$

A. Yes (for every $d \in \mathbb{N}$)

B. No (for every $d \in \mathbb{N}$)

✓ C. Yes in low dimension, no in high dimension

D. Yes in high dimension, no in low dimension



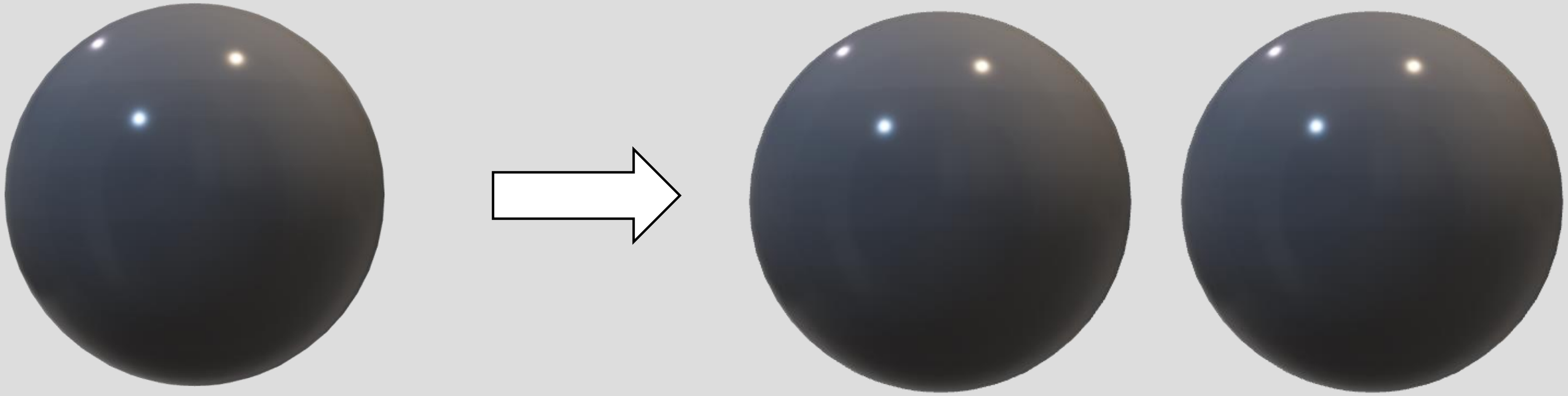
THE BANACH INTEGRAL

$B(\mathbb{R}^d) = \{f : \mathbb{R}^d \to \mathbb{R} : f \text{ is bounded and } \{x \in \mathbb{R}^d : f(x) \neq 0\} \text{ is bounded}\}$

- **Banach integral**: if $d \in \{1,2\}$, then there exists a linear function $L: B(\mathbb{R}^d) \to \mathbb{R}$ such that
  - $L(1_{[0,1)^d}) = 1$
  - For every isometry $\Phi : \mathbb{R}^d \to \mathbb{R}^d, L \circ \Phi = L$
- Applying L to indicator functions → solution to weak problem of measurement
- Proof uses **Hahn—Banach theorem** from functional analysis

THE BANACH—TARSKI PARADOX

- In dimension $d = 3$, there exists a decomposition of the unit ball into finitely many subsets, which can be reassembled (by isometries) into two congruent copies of the unit ball



- Key property: the group of isometries of $\mathbb{R}^3$ contains the free group on two generators and is **non-amenable**

LEBESGUE MEASURE

PROBLEM OF MEASUREMENT IN EUCLIDEAN SPACE

- **Quadrature** (squaring), method of **exhaustion**, and method of **indivisibles** (Cavalieri's principle) all superseded by ***Jordan content***
- Full solution to problem of measurement (assigning measure to every subset of real line/Euclidean space in a consistent way) is impossible*

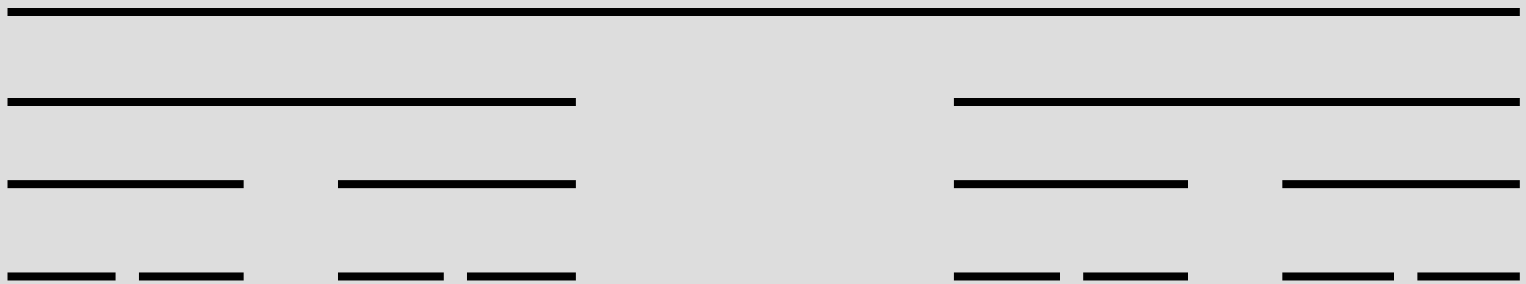
*except in low dimension if we sacrifice countable additivity for finite additivity instead

- Can we do better than Jordan content?

DRAWBACK OF JORDAN CONTENT: NOT COUNTABLY ADDITIVE

Smith—Volterra—Cantor set K:

set remaining after successively removing middle intervals of length 4^{-n} at step n



- Complement $U = [0,1] \setminus K$ is a countable union of open intervals, contributing length $2^{n-1} \cdot 4^{-n} = 4 \cdot 2^{-(n-1)}$ in step n
- $J_*(U) = \sum_{n=1}^{\infty} 4 \cdot 2^{-(n-1)} = \frac{1}{2}$
- K does not contain any intervals, so U is dense → $J^*(U) = 1$.

SUPPOSE $f_n : [0,1] \to [0,1]$ IS RIEMANN-INTEGRABLE FOR EACH $n \in \mathbb{N}$ AND $f(x) = \lim_{n\to\infty} f_n(x)$ EXISTS FOR EVERY $x \in [0,1]$. TRUE OR FALSE: f IS RIEMANN-INTEGRABLE AND $\int_0^1 f(x)\,dx = \lim_{n\to\infty} \int_0^1 f_n(x)\,dx$.

A. True

✓ B. False

0	0
A	B

LEBESGUE MEASURE

- Jordan content approximates objects by finite unions of boxes
- Lebesgue measure approximates objects by **countable unions** of boxes
- There exists a family of sets $\mathcal{M} \subseteq \mathcal{P}(\mathbb{R}^d)$ that contains
  - All Jordan-measurable sets
  - All open and closed subsets of $\mathbb{R}^d$
  - Complements and countable unions/intersections of elements of $\mathcal{M}$
- The **Lebesgue measure**

$$\lambda(E) = \inf\left\{\sum_{n=1}^{\infty} \mathrm{Vol}(B_n) : B_n \text{ are boxes and } E \subseteq \bigcup_{n=1}^{\infty} B_n\right\}$$

is countably additive and translation-invariant on $\mathcal{M}$

CONVERGENCE THEOREMS

The Lebesgue measure λ has an accompanying theory of integration satisfying:

- **Monotone convergence theorem**: if $f_1 \leq f_2 \leq \cdots$ are "measurable" and $f(x) = \lim_{n\to\infty} f_n(x)$, then $\int f = \lim_{n\to\infty} \int f_n$
- **Bounded convergence theorem**: if $f_n: [0,1)^d \to [0,1]$ are measurable and $f(x) = \lim_{n\to\infty} f_n(x)$ exists for every $x \in [0,1)^d$, then$\int f = \lim_{n\to\infty} \int f_n$

ABSTRACT MEASURES

AXIOMATIC APPROACH TO MEASURE

- Space X
- Collection of "measurable subsets" of X playing the role of "geometric objects"
- Countably additive function ("measure") assigning a size to each measurable subset

APPLICATIONS OF MEASURE THEORY

PROBABILITY THEORY

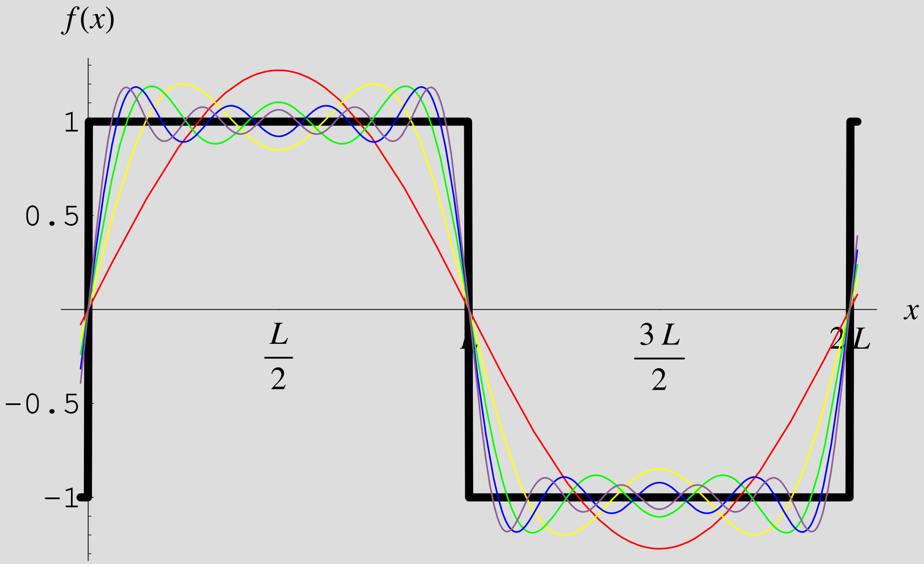
- Mathematical formalism for probability
- Rigorous defintions for **random variables**
- Limit theorems (e.g., law of large numbers, central limit theorem) are measure-theoretic convergence results
- Abstract measure theory unifies **discrete** and **continuous** realms

HARMONIC ANALYSIS

- Periodic continuous functions on $\mathbb{R}$ decompose as Fourier series

$$f(x) \sim \sum_{n\in\mathbb{Z}} \hat{f}(n)\, e^{2\pi i n x}$$

- Several questions about this series require measure theory to answer:
  - Which sequences $(a_n)_{n\in\mathbb{Z}}$ arise as Fourier coefficients $a_n = \hat{f}(n)$?
  - Does the Fourier series converge to f? In which sense?
  - Are there similar decompositions for functions defined on other groups?



FUNCTIONAL ANALYSIS AND OPERATOR THEORY

- Familiar notions from linear algebra require measures to extend to infinite-dimensional spaces
- Unitary matrices can be diagonalized in the form

$$U = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix}$$

with $|\lambda_i| = 1$

- Infinite-dimensional version for unitary operators represents spectrum (“eigenvalues”) as a measure on the unit circle

ERGODIC THEORY

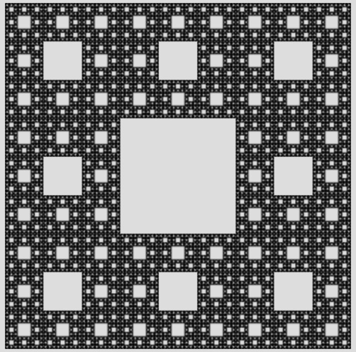
- Study of long-term statistical behavior of dynamical systems
- Originally motivated by "ergodic hypothesis" in statistical thermodynamics and stability questions about the solar system
- Formalized by ***measure-preserving transformations*** on abstract measure spaces

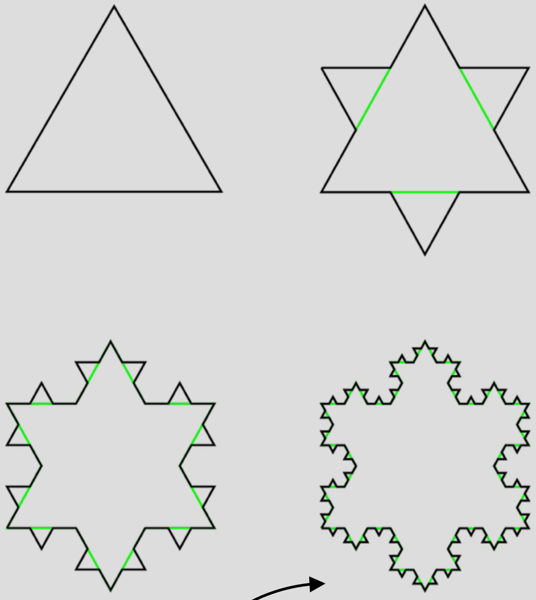
FRACTAL GEOMETRY

- Self-similar objects can have sensible notion of dimension that is non-integer
- Formalized by constructing ***Hausdorff measures***, which interpolate between integer-dimensional Lebesgue measures

Sierpiński carpet

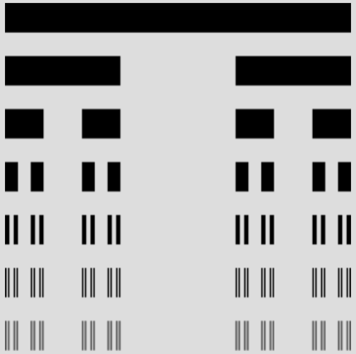
Dimension: $\frac{\log 8}{\log 3} \approx 1.89$





Koch snowflake

Dimension: $\frac{\log 4}{\log 3} \approx 1.26$



Cantor set

Dimension: $\frac{\log 2}{\log 3} \approx 0.63$